In this chapter we develop the forward or configuration kinematic equations for rigid robots. The forward kinematics problem is concerned with the relationship between the individual joints of the robot manipulator and the position and orientation of the tool or end-effector. Stated more formally, the forward kinematics problem is to determine the position and orientation of the end-effector, given the values for the joint variables of the robot. The joint variables are the angles between the links in the case of revolute or rotational joints, and the link extension in the case of prismatic or sliding joints. The forward kinematics problem is to be contrasted with the inverse kinematics problem, which will be studied in the next chapter, and which is concerned with determining values for the joint variables that achieve a desired position and orientation for the end-effector of the robot.

3.1 Kinematic Chains

As described in Chapter 1, a robot manipulator is composed of a set of links connected together by various joints. The joints can either be very simple, such as a revolute joint or a prismatic joint, or else they can be more complex, such as a ball and socket joint. (Recall that a revolute joint is like a hinge and allows a relative rotation about a single axis, and a prismatic joint permits a linear motion along a single axis, namely an extension or retraction.) The difference between the two situations is that, in the first instance, the joint has only a single degree-of-freedom of motion: the angle of rotation in the case of a revolute joint, and the amount of linear displacement in the case of a prismatic joint. In contrast, a ball and socket joint has two degrees-of-freedom. In this book it is assumed throughout that all joints have only a single degree-of-freedom. Note that the assumption
does not involve any real loss of generality, since joints such as a ball and socket joint (two
degrees-of-freedom) or a spherical wrist (three degrees-of-freedom) can always be thought
of as a succession of single degree-of-freedom joints with links of length zero in between.

With the assumption that each joint has a single degree-of-freedom, the action of each
joint can be described by a single real number: the angle of rotation in the case of a rev-
olute joint or the displacement in the case of a prismatic joint. The objective of forward
kinematic analysis is to determine the cumulative effect of the entire set of joint variables.
In this chapter we will develop a set of conventions that provide a systematic procedure for
performing this analysis. It is, of course, possible to carry out forward kinematics analysis
even without respecting these conventions, as we did for the two-link planar manipulator
example in Chapter 1. However, the kinematic analysis of an \( n \)-link manipulator can be
extremely complex and the conventions introduced below simplify the analysis consider-
ably. Moreover, they give rise to a universal language with which robot engineers can
communicate.

A robot manipulator with \( n \) joints will have \( n + 1 \) links, since each joint connects two
links. We number the joints from 1 to \( n \), and we number the links from 0 to \( n \), starting
from the base. By this convention, joint \( i \) connects link \( i - 1 \) to link \( i \). We will consider
the location of joint \( i \) to be fixed with respect to link \( i - 1 \). When joint \( i \) is actuated, link
\( i \) moves. Therefore, link 0 (the first link) is fixed, and does not move when the joints are
actuated. Of course the robot manipulator could itself be mobile (e.g., it could be mounted
on a mobile platform or on an autonomous vehicle), but we will not consider this case in
the present chapter, since it can be handled easily by slightly extending the techniques
presented here.

With the \( i^{th} \) joint, we associate a joint variable, denoted by \( q_i \). In the case of a rev-
olute joint, \( q_i \) is the angle of rotation, and in the case of a prismatic joint, \( q_i \) is the joint
displacement:

\[
q_i = \begin{cases} 
\theta_i : & \text{joint i revolute} \\
 d_i : & \text{joint i prismatic}
\end{cases}
\]

To perform the kinematic analysis, we rigidly attach a coordinate frame to each link.
In particular, we attach \( o_i x_i y_i z_i \) to link \( i \). This means that, whatever motion the robot
executes, the coordinates of each point on link \( i \) are constant when expressed in the \( i^{th} \)
coordinate frame. Furthermore, when joint \( i \) is actuated, link \( i \) and its attached frame,
\( o_i x_i y_i z_i \), experience a resulting motion. The frame \( o_0 x_0 y_0 z_0 \), which is attached to the robot
base, is referred to as the inertial frame. Figure 3.1 illustrates the idea of attaching frames
rigidly to links in the case of an elbow manipulator.

Now suppose \( A_i \) is the homogeneous transformation matrix that expresses the position
and orientation of \( o_i x_i y_i z_i \) with respect to \( o_{i-1} x_{i-1} y_{i-1} z_{i-1} \). The matrix \( A_i \) is not constant,
but varies as the configuration of the robot is changed. However, the assumption that all
joints are either revolute or prismatic means that \( A_i \) is a function of only a single joint
variable, namely \( q_i \). In other words,

\[
A_i = A_i(q_i).
\]
3.1. KINEMATIC CHAINS

Figure 3.1: Coordinate frames attached to elbow manipulator.

Now the homogeneous transformation matrix that expresses the position and orientation of $o_jx_jy_jz_j$ with respect to $o_ix_iy_iz_i$ is called, by convention, a transformation matrix, and is denoted by $T^i_j$. From Chapter 2 we see that

$$T^i_j = \begin{cases} A_{i+1}A_{i+2} \ldots A_{j-1}A_j & \text{if } i < j \\ I & \text{if } i = j \\ (T^i_j)^{-1} & \text{if } j > i. \end{cases} \quad (3.3)$$

By the manner in which we have rigidly attached the various frames to the corresponding links, it follows that the position of any point on the end-effector, when expressed in frame $n$, is a constant independent of the configuration of the robot. Denote the position and orientation of the end-effector with respect to the inertial or base frame by a three-vector $o_0^n$ (which gives the coordinates of the origin of the end-effector frame with respect to the base frame) and the $3 \times 3$ rotation matrix $R_0^n$, and define the homogeneous transformation matrix

$$H = \begin{bmatrix} R_0^n & o_0^n \\ 0 & 1 \end{bmatrix}. \quad (3.4)$$

Then the position and orientation of the end-effector in the inertial frame are given by

$$H = T^n_o = A_1(q_1) \cdots A_n(q_n). \quad (3.5)$$

Each homogeneous transformation $A_i$ is of the form

$$A_i = \begin{bmatrix} R_i^{-1} & o_i^{-1} \\ 0 & 1 \end{bmatrix}. \quad (3.6)$$

Hence

$$T^i_j = A_{i+1} \cdots A_j = \begin{bmatrix} R^i_j & o^i_j \\ 0 & 1 \end{bmatrix}. \quad (3.7)$$
The matrix $R_i$ expresses the orientation of $o_jx_jy_jz_j$ relative to $o_ix_iy_iz_i$ and is given by the rotational parts of the $A$-matrices as

$$R_i = R_{i+1} \cdots R_{j-1}. \quad (3.8)$$

The coordinate vectors $o_j$ are given recursively by the formula

$$o_j = o_{j-1} + R_{i-1}o_{j-1}. \quad (3.9)$$

These expressions will be useful in Chapter 5 when we study Jacobian matrices.

In principle, that is all there is to forward kinematics! Determine the functions $A_i(q_i)$, and multiply them together as needed. However, it is possible to achieve a considerable amount of streamlining and simplification by introducing further conventions, such as the Denavit-Hartenberg representation of a joint, and this is the objective of the remainder of the chapter.

### 3.2 Denavit Hartenberg Representation

While it is possible to carry out all of the analysis in this chapter using an arbitrary frame attached to each link, it is helpful to be systematic in the choice of these frames. A commonly used convention for selecting frames of reference in robotic applications is the Denavit-Hartenberg, or D-H convention. In this convention, each homogeneous transformation $A_i$ is represented as a product of four basic transformations

$$A_i = \text{Rot}_{z,\theta_i} \text{Trans}_{z, d_i} \text{Trans}_{x, a_i} \text{Rot}_{x, \alpha_i} \quad (3.10)$$


$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where the four quantities $\theta_i$, $a_i$, $d_i$, $\alpha_i$ are parameters associated with link $i$ and joint $i$. The four parameters $a_i$, $\alpha_i$, $d_i$, and $\theta_i$ in (3.10) are generally given the names **link length**, **link twist**, **link offset**, and **joint angle**, respectively. These names derive from specific aspects of the geometric relationship between two coordinate frames, as will become apparent below. Since the matrix $A_i$ is a function of a single variable, it turns out that three of the above four quantities are constant for a given link, while the fourth parameter, $\theta_i$ for a revolute joint and $d_i$ for a prismatic joint, is the joint variable.

From Chapter 2 one can see that an arbitrary homogeneous transformation matrix can be characterized by six numbers, such as, for example, three numbers to specify the...
3.2. DENAVIT HARTENBERG REPRESENTATION

The fourth column of the matrix and three Euler angles to specify the upper left $3 \times 3$ rotation matrix. In the D-H representation, in contrast, there are only four parameters. How is this possible? The answer is that, while frame $i$ is required to be rigidly attached to link $i$, we have considerable freedom in choosing the origin and the coordinate axes of the frame. For example, it is not necessary that the origin, $o_i$, of frame $i$ be placed at the physical end of link $i$. In fact, it is not even necessary that frame $i$ be placed within the physical link; frame $i$ could lie in free space — so long as frame $i$ is rigidly attached to link $i$. By a clever choice of the origin and the coordinate axes, it is possible to cut down the number of parameters needed from six to four (or even fewer in some cases). In Section 3.2.1 we will show why, and under what conditions, this can be done, and in Section 3.2.2 we will show exactly how to make the coordinate frame assignments.

3.2.1 Existence and uniqueness issues

Clearly it is not possible to represent any arbitrary homogeneous transformation using only four parameters. Therefore, we begin by determining just which homogeneous transformations can be expressed in the form (3.10). Suppose we are given two frames, denoted by frames 0 and 1, respectively. Then there exists a unique homogeneous transformation matrix $A$ that takes the coordinates from frame 1 into those of frame 0. Now suppose the two frames have two additional features, namely:

(DH1) The axis $x_1$ is perpendicular to the axis $z_0$

(DH2) The axis $x_1$ intersects the axis $z_0$

as shown in Figure 3.2. Under these conditions, we claim that there exist unique numbers $a$, $d$, $\theta$, $\alpha$ such that

$$A = \text{Rot}_{z,\theta} \text{Trans}_{z,d} \text{Trans}_{x,a} \text{Rot}_{x,\alpha}. \quad (3.11)$$
Of course, since $\theta$ and $\alpha$ are angles, we really mean that they are unique to within a multiple of $2\pi$. To show that the matrix $A$ can be written in this form, write $A$ as

$$A = \begin{bmatrix} R^0_1 & o^0_1 \\ 0 & 1 \end{bmatrix} \quad (3.12)$$

and let $r_i$ denote the $i^{th}$ column of the rotation matrix $R^0_1$. We will now examine the implications of the two DH constraints.

If (DH1) is satisfied, then $x_1$ is perpendicular to $z_0$ and we have $x_1 \cdot z_0 = 0$. Expressing this constraint with respect to $o_0x_0y_0z_0$, using the fact that $r_1$ is the representation of the unit vector $x_1$ with respect to frame 0, we obtain

$$0 = x^0_1 \cdot z^0_0 \quad (3.13)$$

Expressing this constraint with respect to $o_0x_0y_0z_0$, using the fact that $r_1$ is the representation of the unit vector $x_1$ with respect to frame 0, we obtain

$$0 = x^0_1 \cdot z^0_0 = [r_{11}, r_{21}, r_{31}]^T \cdot [0, 0, 1]^T \quad (3.14)$$

$$= r_{31}. \quad (3.15)$$

Since $r_{31} = 0$, we now need only show that there exist unique angles $\theta$ and $\alpha$ such that

$$R^0_1 = R_{x,\theta}R_{z,\alpha} = \begin{bmatrix} c\theta & -s\theta c\alpha & s\theta s\alpha \\ s\theta & c\theta c\alpha & -c\theta s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}. \quad (3.16)$$

The only information we have is that $r_{31} = 0$, but this is enough. First, since each row and column of $R^0_1$ must have unit length, $r_{31} = 0$ implies that

$$r^2_{11} + r^2_{21} = 1,$$

$$r^2_{32} + r^2_{33} = 1. \quad (3.17)$$

Hence there exist unique $\theta$, $\alpha$ such that

$$(r_{11}, r_{21}) = (c\theta, s\theta), \quad (r_{33}, r_{32}) = (c\alpha, s\alpha). \quad (3.18)$$

Once $\theta$ and $\alpha$ are found, it is routine to show that the remaining elements of $R^0_1$ must have the form shown in (3.16), using the fact that $R^0_1$ is a rotation matrix.

Next, assumption (DH2) means that the displacement between $o_0$ and $o_1$ can be expressed as a linear combination of the vectors $z_0$ and $x_1$. This can be written as $o_1 = o_0 + dz_0 + ax_1$. Again, we can express this relationship in the coordinates of $o_0x_0y_0z_0$, and we obtain

$$o^0_1 = o^0_0 + dz^0_0 + ax^0_1 \quad (3.19)$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} \quad (3.20)$$

$$= \begin{bmatrix} ac\theta \\ as\theta \\ d \end{bmatrix}. \quad (3.21)$$
Combining the above results, we obtain (3.10) as claimed. Thus, we see that four parameters are sufficient to specify any homogeneous transformation that satisfies the constraints (DH1) and (DH2).

Now that we have established that each homogeneous transformation matrix satisfying conditions (DH1) and (DH2) above can be represented in the form (3.10), we can in fact give a physical interpretation to each of the four quantities in (3.10). The parameter $a$ is the distance between the axes $z_0$ and $z_1$, and is measured along the axis $x_1$. The angle $\alpha$ is the angle between the axes $z_0$ and $z_1$, measured in a plane normal to $x_1$. The positive sense for $\alpha$ is determined from $z_0$ to $z_1$ by the right-hand rule as shown in Figure 3.3. The parameter $d$ is the distance between the origin $o_0$ and the intersection of the $x_1$ axis with $z_0$ measured along the $z_0$ axis. Finally, $\theta$ is the angle between $x_0$ and $x_1$ measured in a plane normal to $z_0$. These physical interpretations will prove useful in developing a procedure for assigning coordinate frames that satisfy the constraints (DH1) and (DH2), and we now turn our attention to developing such a procedure.

### 3.2.2 Assigning the coordinate frames

For a given robot manipulator, one can always choose the frames $0, \ldots, n$ in such a way that the above two conditions are satisfied. In certain circumstances, this will require placing the origin $o_i$ of frame $i$ in a location that may not be intuitively satisfying, but typically this will not be the case. In reading the material below, it is important to keep in mind that the choices of the various coordinate frames are not unique, even when constrained by the requirements above. Thus, it is possible that different engineers will derive differing, but equally correct, coordinate frame assignments for the links of the robot. It is very important to note, however, that the end result (i.e., the matrix $T^0_n$) will be the same, regardless of the assignment of intermediate link frames (assuming that the coordinate frames for link $n$ coincide). We will begin by deriving the general procedure. We will then discuss various common special cases where it is possible to further simplify the homogeneous transformation matrix.
To start, note that the choice of $z_i$ is arbitrary. In particular, from (3.16), we see that by choosing $\alpha_i$ and $\theta_i$ appropriately, we can obtain any arbitrary direction for $z_i$. Thus, for our first step, we assign the axes $z_0, \ldots, z_{n-1}$ in an intuitively pleasing fashion. Specifically, we assign $z_i$ to be the axis of actuation for joint $i + 1$. Thus, $z_0$ is the axis of actuation for joint 1, $z_1$ is the axis of actuation for joint 2, etc. There are two cases to consider: (i) if joint $i + 1$ is revolute, $z_i$ is the axis of revolution of joint $i + 1$; (ii) if joint $i + 1$ is prismatic, $z_i$ is the axis of translation of joint $i + 1$. At first it may seem a bit confusing to associate $z_i$ with joint $i + 1$, but recall that this satisfies the convention that we established in Section 3.1, namely that joint $i$ is fixed with respect to frame $i$, and that when joint $i$ is actuated, link $i$ and its attached frame, $o_i x_i y_i z_i$, experience a resulting motion.

Once we have established the $z$-axes for the links, we establish the base frame. The choice of a base frame is nearly arbitrary. We may choose the origin $o_0$ of the base frame to be any point on $z_0$. We then choose $x_0, y_0$ in any convenient manner so long as the resulting frame is right-handed. This sets up frame 0.

Once frame 0 has been established, we begin an iterative process in which we define frame $i$ using frame $i - 1$, beginning with frame 1. Figure 3.4 will be useful for understanding the process that we now describe.

![Figure 3.4: Denavit-Hartenberg frame assignment.](image)

In order to set up frame $i$ it is necessary to consider three cases: (i) the axes $z_{i-1}, z_i$ are not coplanar, (ii) the axes $z_{i-1}, z_i$ intersect (iii) the axes $z_{i-1}, z_i$ are parallel. Note that in both cases (ii) and (iii) the axes $z_{i-1}$ and $z_i$ are coplanar. This situation is in fact quite common, as we will see in Section 3.3. We now consider each of these three cases.

1. **$z_{i-1}$ and $z_i$ are not coplanar:** If $z_{i-1}$ and $z_i$ are not coplanar, then there exists a unique line segment perpendicular to both $z_{i-1}$ and $z_i$ such that it connects both lines and it has minimum length. The line containing this common normal to $z_{i-1}$ and $z_i$ defines $x_i$,
and the point where this line intersects $z_i$ is the origin $o_i$. By construction, both conditions (DH1) and (DH2) are satisfied and the vector from $o_{i-1}$ to $o_i$ is a linear combination of $z_{i-1}$ and $x_i$. The specification of frame $i$ is completed by choosing the axis $y_i$ to form a right-hand frame. Since assumptions (DH1) and (DH2) are satisfied the homogeneous transformation matrix $A_i$ is of the form (3.10).

(ii) $z_{i-1}$ is parallel to $z_i$: If the axes $z_{i-1}$ and $z_i$ are parallel, then there are infinitely many common normals between them and condition (DH1) does not specify $x_i$ completely. In this case we are free to choose the origin $o_i$ anywhere along $z_i$. One often chooses $o_i$ to simplify the resulting equations. The axis $x_i$ is then chosen either to be directed from $o_i$ toward $z_{i-1}$, along the common normal, or as the opposite of this vector. A common method for choosing $o_i$ is to choose the normal that passes through $o_{i-1}$ as the $x_i$ axis; $o_i$ is then the point at which this normal intersects $z_i$. In this case, $d_i$ would be equal to zero. Once $x_i$ is fixed, $y_i$ is determined, as usual by the right hand rule. Since the axes $z_{i-1}$ and $z_i$ are parallel, $\alpha_i$ will be zero in this case.

(iii) $z_{i-1}$ intersects $z_i$: In this case $x_i$ is chosen normal to the plane formed by $z_i$ and $z_{i-1}$. The positive direction of $x_i$ is arbitrary. The most natural choice for the origin $o_i$ in this case is at the point of intersection of $z_i$ and $z_{i-1}$. However, any convenient point along the axis $z_i$ suffices. Note that in this case the parameter $a_i$ equals 0.

This constructive procedure works for frames $0, \ldots, n-1$ in an $n$-link robot. To complete the construction, it is necessary to specify frame $n$. The final coordinate system $o_nx_ny_nz_n$ is commonly referred to as the end-effector or tool frame (see Figure 3.5). The origin $o_n$ is most often placed symmetrically between the fingers of the gripper. The unit vectors along the $x_n$, $y_n$, and $z_n$ axes are labeled as $n$, $s$, and $a$, respectively. The terminology arises from fact that the direction $a$ is the approach direction, in the sense that the gripper typically approaches an object along the $a$ direction. Similarly the $s$ direction is the sliding direction, the direction along which the fingers of the gripper slide to open and close, and $n$ is the direction normal to the plane formed by $a$ and $s$. 

![Figure 3.5: Tool frame assignment.](image)
In contemporary robots the final joint motion is a rotation of the end-effector by $\theta_n$ and the final two joint axes, $z_{n-1}$ and $z_n$, coincide. In this case, the transformation between the final two coordinate frames is a translation along $z_{n-1}$ by a distance $d_n$ followed (or preceded) by a rotation of $\theta_n$ radians about $z_{n-1}$. This is an important observation that will simplify the computation of the inverse kinematics in the next chapter.

Finally, note the following important fact. In all cases, whether the joint in question is revolute or prismatic, the quantities $a_i$ and $\alpha_i$ are always constant for all $i$ and are characteristic of the manipulator. If joint $i$ is prismatic, then $\theta_i$ is also a constant, while $d_i$ is the $i^{th}$ joint variable. Similarly, if joint $i$ is revolute, then $d_i$ is constant and $\theta_i$ is the $i^{th}$ joint variable.

### 3.2.3 Summary

We may summarize the above procedure based on the D-H convention in the following algorithm for deriving the forward kinematics for any manipulator.

**Step 1:** Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

**Step 2:** Establish the base frame. Set the origin anywhere on the $z_0$-axis. The $x_0$ and $y_0$ axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

**Step 3:** Locate the origin $o_i$ where the common normal to $z_i$ and $z_{i-1}$ intersects $z_i$. If $z_i$ intersects $z_{i-1}$ locate $o_i$ at this intersection. If $z_i$ and $z_{i-1}$ are parallel, locate $o_i$ in any convenient position along $z_i$.

**Step 4:** Establish $x_i$ along the common normal between $z_{i-1}$ and $z_i$ through $o_i$, or in the direction normal to the $z_{i-1} - z_i$ plane if $z_{i-1}$ and $z_i$ intersect.

**Step 5:** Establish $y_i$ to complete a right-hand frame.

**Step 6:** Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the $n$-th joint is revolute, set $z_n = a$ along the direction $z_{n-1}$. Establish the origin $o_n$ conveniently along $z_n$, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = s$ in the direction of the gripper closure and set $x_n = n$ as $s \times a$. If the tool is not a simple gripper set $x_n$ and $y_n$ conveniently to form a right-hand frame.

**Step 7:** Create a table of link parameters $a_i$, $d_i$, $\alpha_i$, $\theta_i$.

- $a_i = \text{distance along } x_i \text{ from } o_i \text{ to the intersection of the } x_i \text{ and } z_{i-1} \text{ axes.}$
- $d_i = \text{distance along } z_{i-1} \text{ from } o_{i-1} \text{ to the intersection of the } x_i \text{ and } z_{i-1} \text{ axes. } d_i \text{ is }$
- $\alpha_i = \text{variable if joint } i \text{ is prismatic.}$
- $\alpha_i = \text{the angle between } z_{i-1} \text{ and } z_i \text{ measured about } x_i \text{ (see Figure 3.3).}$
\[ \theta_i = \text{the angle between } x_{i-1} \text{ and } x_i \text{ measured about } z_{i-1} \text{ (see Figure 3.3). } \theta_i \text{ is variable if joint } i \text{ is revolute.} \]

**Step 8:** Form the homogeneous transformation matrices \( A_i \) by substituting the above parameters into (3.10).

**Step 9:** Form \( T^n_0 = A_1 \cdots A_n \). This then gives the position and orientation of the tool frame expressed in base coordinates.

### 3.3 Examples

In the D-H convention the only variable angle is \( \theta \), so we simplify notation by writing \( c_i \) for \( \cos \theta_i \), etc. We also denote \( \theta_1 + \theta_2 \) by \( \theta_{12} \), and \( \cos(\theta_1 + \theta_2) \) by \( c_{12} \), and so on. In the following examples it is important to remember that the D-H convention, while systematic, still allows considerable freedom in the choice of some of the manipulator parameters. This is particularly true in the case of parallel joint axes or when prismatic joints are involved.

**Example 3.1 Planar Elbow Manipulator**

Consider the two-link planar arm of Figure 3.6. The joint axes \( z_0 \) and \( z_1 \) are normal to the page. We establish the base frame \( o_0x_0y_0z_0 \) as shown. The origin is chosen at the point of intersection of the \( z_0 \) axis with the page and the direction of the \( x_0 \) axis is completely arbitrary. Once the base frame is established, the \( o_1x_1y_1z_1 \) frame is fixed as shown by the D-H convention, where the origin \( o_1 \) has been located at the intersection of \( z_1 \) and the page. The final frame \( o_2x_2y_2z_2 \) is fixed by choosing the origin \( o_2 \) at the end of link 2 as shown. The link parameters are shown in Table 3.1. The \( A \)-matrices are determined from (3.10) as

![Figure 3.6: Two-link planar manipulator. The z-axes all point out of the page, and are not shown in the figure.](image)
Table 3.1: Link parameters for 2-link planar manipulator.

<table>
<thead>
<tr>
<th>Link</th>
<th>$a_i$</th>
<th>$\alpha_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_1$</td>
<td>0</td>
<td>0</td>
<td>$\theta^*_1$</td>
</tr>
<tr>
<td>2</td>
<td>$a_2$</td>
<td>0</td>
<td>0</td>
<td>$\theta^*_2$</td>
</tr>
</tbody>
</table>

* variable

\[
A_1 = \begin{bmatrix}
    c_1 & -s_1 & 0 & a_1 c_1 \\
    s_1 & c_1 & 0 & a_1 s_1 \\
    0   & 0   & 1 & 0 \\
    0   & 0   & 0 & 1
\end{bmatrix}.
\]  

(3.22)

\[
A_2 = \begin{bmatrix}
    c_2 & -s_2 & 0 & a_2 c_2 \\
    s_2 & c_2 & 0 & a_2 s_2 \\
    0   & 0   & 1 & 0 \\
    0   & 0   & 0 & 1
\end{bmatrix}.
\]  

(3.23)

The $T$-matrices are thus given by

\[
T^o_i = A_i.
\]  

(3.24)

\[
T^o_2 = A_1 A_2 = \begin{bmatrix}
    c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\
    s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\
    0   & 0   & 1 & 0 \\
    0   & 0   & 0 & 1
\end{bmatrix}.
\]  

(3.25)

Notice that the first two entries of the last column of $T^o_2$ are the $x$ and $y$ components of the origin $o_2$ in the base frame; that is,

\[
x = a_1 c_1 + a_2 c_{12}
\]  

(3.26)

\[
y = a_1 s_1 + a_2 s_{12}
\]

are the coordinates of the end-effector in the base frame. The rotational part of $T^o_2$ gives the orientation of the frame $o_2 x_2 y_2 z_2$ relative to the base frame.

Example 3.2 Three-Link Cylindrical Robot

Consider now the three-link cylindrical robot represented symbolically by Figure 3.7. We establish $o_0$ as shown at joint 1. Note that the placement of the origin $o_0$ along $z_0$ as well as the direction of the $x_0$ axis are arbitrary. Our choice of $o_0$ is the most natural, but $o_0$ could just as well be placed at joint 2. The axis $x_0$ is chosen normal to the page. Next, since $z_0$ and $z_1$ coincide, the origin $o_1$ is chosen at joint 1 as shown. The $x_1$ axis is normal to the page when $\theta_1 = 0$ but, of course its direction will change since $\theta_1$ is variable. Since $z_2$ and $z_1$ intersect, the origin $o_2$ is placed at this intersection. The direction of $x_2$ is chosen
3.3. EXAMPLES

Figure 3.7: Three-link cylindrical manipulator.

Table 3.2: Link parameters for 3-link cylindrical manipulator.

<table>
<thead>
<tr>
<th>Link</th>
<th>$a_i$</th>
<th>$\alpha_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$d_1$</td>
<td>$\theta_1^*$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$-90$</td>
<td>$d_2$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>$d_3$</td>
<td>0</td>
</tr>
</tbody>
</table>

* variable

parallel to $x_1$ so that $\theta_2$ is zero. Finally, the third frame is chosen at the end of link 3 as shown.

The link parameters are now shown in Table 3.2. The corresponding $A$ and $T$ matrices
are

\[ A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hspace{1cm} (3.27)\

\[ A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ T_3^o = A_1A_2A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1d_3 \\ s_1 & 0 & c_1 & c_1d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]  \hspace{1cm} (3.28)

\[ \Box \]

**Example 3.3 Spherical Wrist**

![Figure 3.8: The spherical wrist frame assignment.](image)

The spherical wrist configuration is shown in Figure 3.8, in which the joint axes \( z_3, z_4, z_5 \) intersect at \( o \). The Denavit-Hartenberg parameters are shown in Table 3.3. The Stanford manipulator is an example of a manipulator that possesses a wrist of this type. In fact, the following analysis applies to virtually all spherical wrists.

We show now that the final three joint variables, \( \theta_4, \theta_5, \theta_6 \) are the Euler angles \( \phi, \theta, \psi \), respectively, with respect to the coordinate frame \( o_3x_3y_3z_3 \). To see this we need only compute